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Porosity of mutually nearest and mutually furthest points in Banach spaces ☆

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Abstract

Let X be a real strictly convex and Kadec Banach space and G a nonempty closed relatively boundedly weakly compact subset of X. Let $\mathscr{B}(X)$ (resp. $\mathscr{K}(X)$) be the family of nonempty bounded closed (resp. compact) subsets of X endowed with the Hausdorff distance and let $\mathscr{B}_G(X)$ denote the closure of the set $\{A \in \mathscr{B}(X) : A \cap G = \emptyset\}$ and $\mathscr{K}_G(X) = \mathscr{B}_G(X) \cap \mathscr{K}(X)$. We introduce the admissible family $\mathscr{A}$ of $\mathscr{B}(X)$ and prove that $E^o_{\mathscr{A}}(G)$ (resp. $E^{\mathscr{A}}_o(G)$), the set of all subsets $F \in \mathscr{A} \subseteq \mathscr{B}_G(X)$ (resp. $F \in \mathscr{A} \subseteq \mathscr{B}(X)$) such that the minimization problem $\min(F, G)$ (resp. the maximization problem $\max(F, G)$) is well-posed, is a dense G_δ-subset of $\mathscr{A}$. Furthermore, when X is uniformly convex, we prove that $\mathscr{A} \backslash E^o_{\mathscr{A}}(G)$ and $\mathscr{A} \backslash E^{\mathscr{A}}_o(G)$ are σ-porous in $\mathscr{A}$.

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1. Introduction

Let X be a real Banach space. We denote by $\mathscr{B}(X)$ the family of nonempty closed bounded subsets of X and $\mathscr{K}(X)$ the family of nonempty compact subsets of X. For a closed subset G of X and $A \in \mathscr{B}(X)$, we set

$$\lambda_{AG} := \inf\{||z - x|| : x \in A, z \in G\}$$

and if G is bounded,

$$\mu_{AG} := \sup\{||z - x|| : x \in A, z \in G\}.$$

Given a nonempty closed (resp. closed bounded) subset G of X, following [5], we say that a pair (x_0, z_0) with $x_0 \in A$ and $z_0 \in G$ is a *solution of the minimization* (resp. *maximization*) problem, denoted by $\min(A, G)$ (resp. $\max(A, G)$), if $||x_0 - z_0|| = \lambda_{AG}$ (resp. $||x_0 - z_0|| = \mu_{AG}$). Moreover, any sequence $\{(x_n, z_n)\}$, where $x_n \in A$ and $z_n \in G$ for all n, such that $\lim_{n\to\infty} ||x_n - z_n|| = \lambda_{AG}$ (resp. $\lim_{n\to\infty} ||x_n - z_n|| = \mu_{AG}$) is called a *minimizing* (resp. *maximizing*) sequence. A minimization (resp. maximization) problem is said to be *well-posed* if it has a unique solution and every minimizing (resp. maximizing) sequence converges strongly to the solution.

Recall that the *Hausdorff distance* on $\mathscr{B}(X)$ is defined by

$$H(A, B) = \max\left\{\sup_{a\in A}\inf_{b\in B} ||a - b||, \sup_{b\in B}\inf_{a\in A} ||a - b||\right\}, \quad A, B \in \mathscr{B}(X).$$

It is well known that $(\mathscr{B}(X), H)$ is a complete metric space.

De Blasi et al. [5] considered the well-posedness of the minimization and maximization problems and set up the generic results for bounded convex closed subsets in a uniformly convex Banach space. Recently, the first author [12] of the present paper established the same results for compact subsets in reflexive locally uniformly convex Banach spaces.

It is the objective of the present paper to further investigate the well-posedness of the mutually nearest and mutually furthest point problems. More precisely, we first introduce the admissible family $\mathscr{A}$ of $\mathscr{B}(X)$ and then establish the generic results on the well-posedness for the admissible family in a strongly convex Banach space. Furthermore, when X is uniformly convex, we prove that the collection of all subsets in the admissible family such that the minimization (respectively, maximization) problems fail to be well-posed is σ-porous in the admissible family. Applying the results to the admissible families $\mathscr{B}(X)$ and $\mathscr{K}(X)$, we immediately extend some recent results due to De Blasi et al. [5] and Li [12].

It should be noted that the problems considered here are also in spirit of Steckin [17] and some further research in this direction can be found in [1,2,6,8–11,13,14] and in the monograph [7]. Some other generic results in spaces of convex sets can be found in [3,5,12,15]. The results of the present paper generalize and sharpen some results from [4,5,12], etc.

2. Preliminaries

In a metric space (E,d), we denote by $B_E(x,r)$ and $U_E(x,r)$ the closed and open ball with center x and radius r, respectively. If X is a Banach space and $A \subset X$, we denote by $\bar{A}$ and diam A the closure and diameter of A, respectively. And we simply write $B(x,r)$ and $U(x,r)$ for $B_X(x,r)$ and $U_X(x,r)$, respectively.

Let $G \subset X$, $F \in \mathscr{B}(X)$ and $x \in X$. We use the notations

$$d(x,G) := \inf\{\|x-g\| : g \in G\},$$

$$L_G(F,\sigma) := \{g \in G :\ d(g,F) \leqslant \lambda_{FG} + \sigma\},$$

where $\sigma > 0$, and if G is bounded,

$$e(x,G) := \sup\{\|x-g\| : g \in G\},$$

$$M_G(F,\sigma) := \{g \in G :\ e(g,F) \geqslant \mu_{FG} - \sigma\}.$$

It is readily seen that $L_G(F,\sigma)$ and $M_G(F,\sigma)$ are nonempty closed and satisfy the property: $L_G(F,\sigma) \subseteq L_G(F,\sigma')$ and $M_G(F,\sigma) \subseteq M_G(F,\sigma')$ if $\sigma < \sigma'$. The next two propositions are straightforward.

Proposition 2.1. *Let $G \subset X$ be a closed subset of X, $F \in \mathscr{B}(X)$ and $x \in X$. Then*

$$\lambda_{FG} \leqslant d(x,F) + d(x,G),$$

and, if G is bounded,

$$\mu_{FG} \geqslant e(x,F) - d(x,G).$$

Proposition 2.2. *Let $G \subset X$ be a closed subset of X and $F \in \mathscr{B}(X)$. If X is uniformly convex, then*

(i) *the problem* $\min(F,G)$ *is well-posed if and only if*

$$\inf_{\sigma>0} \operatorname{diam} L_G(F,\sigma) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \operatorname{diam} L_F(G,\sigma) = 0;$$

(ii) *if G is, in addition, bounded, the problem* $\max(F,G)$ *is well-posed if and only if*

$$\inf_{\sigma>0} \operatorname{diam} M_G(F,\sigma) = 0 \quad \textit{and} \quad \inf_{\sigma>0} \operatorname{diam} M_F(G,\sigma) = 0.$$

The following proposition is useful.

Proposition 2.3. *Assume that X is a uniformly convex Banach and r_0 is a positive real number. Then, for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that, when $0 < \delta \leqslant \delta(\varepsilon)$,*

$$\operatorname{diam} D(x,y,r,\delta) < \varepsilon$$

holds for all $0 < r \leqslant r_0$, $x, y \in X$ *satisfying* $0 < \|x - y\| \leqslant r/2$, *where*

$$D(x, y, r, \delta) = \{z \in X : \|z - y\| \leqslant r - \|x - y\|(1 - \delta) \quad \textit{and} \quad \|z - x\| \geqslant r\}.$$

Proof. Suppose on the contrary that for some $\varepsilon > 0$ and $\forall \delta > 0$, there exist $x^\delta, y^\delta \in X, 0 < r^\delta \leqslant r_0$ satisfying $0 < \|x^\delta - y^\delta\| \leqslant \frac{r^\delta}{2}$ such that

$$\operatorname{diam} D(x^\delta, y^\delta, r^\delta, \delta) > 2\varepsilon. \tag{2.1}$$

With no loss of generality, we may assume $x^\delta = 0$. Write

$$\alpha_\delta := \frac{\|y^\delta\|}{r^\delta}, \quad y_0^\delta := \frac{1}{\alpha_\delta} y^\delta.$$

It is easily seen that $y_0^\delta \in D(0, y^\delta, r^\delta, \delta)$ and $\|y_0^\delta - y^\delta\| = (1 - \alpha_\delta) r^\delta$. We have by (2.1) some $z^\delta \in D(0, y^\delta, r^\delta, \delta)$ such that $\|z^\delta - y_0^\delta\| > \varepsilon$. Hence

$$\|z^\delta\| \geqslant r^\delta \quad \text{and} \quad \|z^\delta - y^\delta\| \leqslant r_\delta - \|y^\delta\|(1 - \delta).$$

From this it is easy to see that

$$\|\alpha_\delta(y^\delta - y_0^\delta) + (1 - \alpha_\delta)(y^\delta - z^\delta)\| = \|(1 - \alpha_\delta) z^\delta\| \geqslant (1 - \alpha_\delta) r^\delta \tag{2.2}$$

and

$$\begin{aligned} (1 - \alpha_\delta) r^\delta &\leqslant \|z^\delta\| - \|y^\delta\| \\ &\leqslant \|z^\delta - y^\delta\| \\ &\leqslant r^\delta - \|y^\delta\|(1 - \delta) \\ &= (1 - \alpha_\delta(1 - \delta)) r^\delta. \end{aligned}$$

By (2.2), we have $x_\delta^* \in X^*, \|x_\delta^*\| = 1$ such that

$$\langle x_\delta^*, \alpha_\delta(y^\delta - y_0^\delta) + (1 - \alpha_\delta)(y^\delta - z^\delta) \rangle \geqslant (1 - \alpha_\delta) r^\delta.$$

It follows that

$$\begin{aligned} \langle x_\delta^*, \alpha_\delta(y^\delta - y_0^\delta) \rangle &\geqslant (1 - \alpha_\delta) r_\delta - (1 - \alpha_\delta)\|y^\delta - z^\delta\| \\ &\geqslant (1 - \alpha_\delta) r_\delta [1 - (1 - \alpha_\delta(1 - \delta))] \\ &= \alpha_\delta (1 - \alpha_\delta)(1 - \delta) r^\delta. \end{aligned}$$

Hence

$$\langle x_\delta^*, y^\delta - y_0^\delta \rangle \geqslant (1 - \alpha_\delta) r^\delta (1 - \delta).$$

Similarly, we also have

$$\langle x_\delta^*, y^\delta - z^\delta \rangle \geqslant (1 - \alpha_\delta) r^\delta.$$

The above two inequalities imply that

$$\|(y^\delta - y_0^\delta) + (y^\delta - z^\delta)\| \geqslant r_\delta (1 - \alpha_\delta)(2 - \delta),$$

which in turn implies that

$$\liminf_{\delta\to 0+}\left\|\frac{y^\delta - y_0^\delta}{r_\delta(1-\alpha_\delta)} + \frac{y^\delta - z^\delta}{r_\delta(1-\alpha_\delta)}\right\| \geqslant 2.$$

Note that

$$\left\|\frac{y^\delta - y_0^\delta}{r_\delta(1-\alpha_\delta)}\right\| = 1$$

and

$$\limsup_{\delta\to 0+}\left\|\frac{y^\delta - z^\delta}{r_\delta(1-\alpha_\delta)}\right\| \leqslant \limsup_{\delta\to 0+}\frac{(1-\alpha_\delta(1-\delta))}{1-\alpha_\delta} = 1$$

as $r_\delta(1-\alpha_\delta)\leqslant r_0/2$. Using the uniform convexity of X, we get

$$\lim_{\delta\to 0+}\|y_0^\delta - z^\delta\| = 0.$$

This contradicts the assumption that $\|z^\delta - y_0^\delta\| > \varepsilon$ and the proof is complete. □

Next we introduce the notion of an admissible family of $\mathscr{B}(X)$.

Definition 2.4. A closed subset $\mathscr{A}$ of $\mathscr{B}(X)$ is called an *admissible family* if, whenever $A\in\mathscr{A}$ and $x\in X$, we have $A\cup\{x\}\in\mathscr{A}$.

Clearly, the most common admissible families are $\mathscr{B}(X)$ and $\mathscr{K}(X)$. Of course, one can construct some other admissible families. For example, let $\mathscr{S}$ be a subset of $\mathscr{B}(X)$. The admissible family spanned by $\mathscr{S}$, denoted $\mathrm{span}_{\mathrm{A}}\mathscr{S}$, is given by

$$\mathrm{span}_{\mathrm{A}}\mathscr{S} = \overline{\{B\cup\{x_1,\ldots,x_n\}\colon\ B\in\mathscr{S},\ n\geqslant 1,\ x_i\in X,\ i=1,\ldots,n\}}.$$

In the rest of this paper, G will be a fixed nonempty closed subset of X. Hence we write, for convenience, $\lambda_F := \lambda_{FG}$ and $\mu_F := \mu_{FG}$ (provided G is also bounded).

3. Existence

Definition 3.1. A Banach space X is said to be (sequentially) *Kadec* if, for each sequence $\{x_n\}\subset X$ which converges weakly to x with $\lim_{n\to\infty}\|x_n\| = \|x\|$, we have $\lim_{n\to\infty}\|x_n - x\| = 0$.

Definition 3.2. A Banach space X is said to be *strongly convex* if it is reflexive, Kadec and strictly convex.

The following results from [6,16] play a key role in the following.

Proposition 3.1. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively boundedly weakly compact (resp. bounded relatively weakly compact) subset of X. Then the set of all points $x\in X$ such that the minimization* *problem* $\min(x, G)$ (*resp. maximization problem* $\max(x, G)$) *is well-posed is a dense* G_δ*-subset of* $X \setminus G$ (*resp.* X).

3.1. The minimization problem

Let $\boldsymbol{N} = \{1, 2, \cdots\}$ and $k \in \boldsymbol{N}$. We use the notations:

- $\mathscr{B}_G(X) := \overline{\{A \in \mathscr{B}(X) : \ \lambda_A > 0\}}$, where the closure is taken in the metric space $(\mathscr{B}(X), H)$.
- $\mathscr{L}_k := \{F \in B_G(X) : \ \inf_{\sigma > 0} \operatorname{diam} L_G(F, \sigma) < \frac{1}{k}$ and $\inf_{\sigma > 0} \operatorname{diam} L_F(G, \sigma) < \frac{1}{k}\}$.
- $E^o_{\mathscr{A}}(G) := \{F \in \mathscr{A}$: the minimization problem $\min(F, G)$ is well-posed$\}$.

In particular, we write $E^o(G)$ for $E^o_{\mathscr{A}}(G)$ when $\mathscr{A} = \mathscr{B}_G(X)$. Repeating the proof of Lemma 3.2 of [5], we have

Lemma 3.1. Let $k \in \boldsymbol{N}$. Then $\mathscr{L}_k$ *is open in* $\mathscr{B}_G(X)$.

Now we are ready to state the first main result of this section.

Theorem 3.1. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively boundedly weakly compact subset of X and* $\mathscr{A} \subseteq \mathscr{B}_G(X)$ *an admissible family of* $\mathscr{B}(X)$. *Then* $E^o_A(G)$ *is a dense* G_δ*-subset of* $\mathscr{A}$.

Proof. By Proposition 2.2 and Lemma 3.1, we see that

$$E^o_{\mathscr{A}}(G) = \bigcap_{k \in \mathbf{N}} (\mathscr{A} \cap \mathscr{L}_k)$$

is a G_δ-subset of $\mathscr{A}$. Thus to complete the proof it suffices to show that $E^o_{\mathscr{A}}(G)$ is dense in $\mathscr{A}$. Towards this end, we take an arbitrary $F \in \mathscr{A}$ and with no loss of generality, we may assume $\lambda_F > 0$. For any $0 < r < \frac{4}{5}\lambda_F$, take $\bar{x} \in F$ such that $d(\bar{x}, G) < \lambda_F + r/4$. By Proposition 3.1, we have an $\tilde{x} \in X$ such that $||\bar{x} - \tilde{x}|| < r/4$ and the minimization problem $\min(\tilde{x}, G)$ is well-posed; hence there is $\tilde{g} \in G$ such that $||\tilde{x} - \tilde{g}|| = d(\tilde{x}, G)$. Set

$$u := \left(1 - \frac{r}{||\tilde{x} - \tilde{g}||}\right)\tilde{x} + \frac{r}{||\tilde{x} - \tilde{g}||}\tilde{g}$$

and

$$Y = F \cup \{u\}.$$

Then we have that $||u - \tilde{x}|| = r$. Since

$$\begin{aligned} ||\tilde{x} - \tilde{g}|| = d(\tilde{x}, G) \geqslant d(\bar{x}, G) - ||\tilde{x} - \bar{x}|| \geqslant \lambda_F \\ - ||\tilde{x} - \bar{x}|| > \lambda_F - \tfrac{r}{4} > r = ||\tilde{x} - u||, \end{aligned} \tag{3.1}$$

the minimization problem $\min(u, G)$ is also well-posed and $\tilde{g}$ is the unique best approximation to u from G. We estimate

$$H(F, Y) = d(u, F) \leqslant ||u - \bar{x}|| \leqslant ||u - \tilde{x}|| + ||\tilde{x} - \bar{x}|| \leqslant \frac{5r}{4}.$$

We next show that $Y \in E^o_{\mathscr{A}}(G)$. Indeed, by (3.1), we obtain

$$\begin{aligned} ||u - \tilde{g}|| &= ||\tilde{x} - \tilde{g}|| - ||\tilde{x} - u|| \\ &\leqslant ||\tilde{x} - \bar{x}|| + d(\bar{x}, G) - r \\ &< r/4 + \lambda_F + r/4 - r \\ &= \lambda_F - r/2. \end{aligned}$$

It follows that

$$\lambda_Y \leqslant ||u - \tilde{g}|| < \lambda_F - r/2.$$

Let now $\{(y_n, g_n)\}$, with $y_n \in Y$ and $g_n \in G$, be a minimizing sequence (i.e. $\lim_n ||y_n - g_n|| = \lambda_Y$). Then,

$$\limsup_n d(y_n, G) \leqslant \lim_n ||y_n - g_n|| = \lambda_Y < \lambda_F - r/2.$$

This implies that there exists some positive integer N_1 such that $y_n \notin F$ and hence $y_n = u$ for all $n \geqslant N_1$. Then we have

$$\lim_n ||g_n - u|| = \lambda_Y \leqslant d(u, G).$$

This shows that $\{g_n\}$ is a minimizing sequence for $\min(u, G)$. Now since $\min(u, G)$ is well-posed, it follows that (g_n) converges strongly to $\tilde{g}$. It is clear that $(u, \tilde{g})$ is the unique solution of the problem $\min(Y, G)$. So $\min(Y, G)$ is well-posed; that is, $Y \in E^o_{\mathscr{A}}(G)$. □

From Theorem 3.1 we immediately have the following corollaries.

Corollary 3.1. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively boundedly weakly compact subset of X. Then $E^o(G)$ is a dense G_δ-subset of $\mathscr{B}_G(X)$.*

Corollary 3.2. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively boundedly weakly compact subset of X. Then $E^o(G) \cap \mathscr{K}(X)$ is a dense G_δ-subset of $\mathscr{K}_G(X)$.*

Corollary 3.3. *Suppose that X is a strongly convex Banach space and G is a nonempty closed subset of X. Let $\mathscr{A} \subseteq \mathscr{B}_G(X)$ be an admissible family of $\mathscr{B}(X)$. Then $E^o_{\mathscr{A}}(G)$ is a dense G_δ-subset of $\mathscr{A}$.*

3.2. The maximization problem

Let $\boldsymbol{N} = \{1, 2, \cdots\}$ and $k \in \boldsymbol{N}$. We use the notations:

- $\mathscr{M}_k := \{F \in \mathscr{B}(X) :\ \inf_{\sigma>0} \operatorname{diam} M_G(F, \sigma) < \frac{1}{k}$ and $\inf_{\sigma>0} \operatorname{diam} M_F(G, \sigma) < \frac{1}{k}\}$.
- $E_o^{\mathscr{A}}(G) := \{F \in \mathscr{A}$: the maximization problem $\max(F, G)$ is well-posed$\}$.

In particular, we write $E_o(G)$ for $E_o^{\mathscr{A}}(G)$ when $\mathscr{A} = \mathscr{B}(X)$. Repeating the proof of Lemma 3.2 of [5], we have

Lemma 3.2. Let $k \in N$. Then $\mathscr{M}_k$ *is open in* $\mathscr{B}(X)$.

The second main result can be stated as follows.

Theorem 3.2. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively weakly compact, and bounded subset of X and* $\mathscr{A}$ *be an admissible family of* $\mathscr{B}(X)$. *Then* $E_o^{\mathscr{A}}(G)$ *is a dense* G_δ*-subset of* $\mathscr{A}$.

Proof. Let $F \in \mathscr{A}$ be arbitrary. Obviously we may assume that $\mu_F > 0$. For any $0 < r < \mu_F$, take $\bar{x} \in F$ such that $e(\bar{x}, G) > \mu_F - r/4$. By Proposition 3.1, there exists $\tilde{x} \in X$ such that $||\bar{x} - \tilde{x}|| < r/4$ and the maximization problem $\max(\tilde{x}, G)$ is well-posed. Let $\tilde{g} \in G$ with $||\tilde{x} - \tilde{g}|| = e(\tilde{x}, G)$ and set

$$u := \left(1 + \frac{r}{||\tilde{x} - \tilde{g}||}\right)\tilde{x} - \frac{r}{||\tilde{x} - \tilde{g}||}\tilde{g}$$

and

$$Y := F \cup \{u\}.$$

Then we have $||u - \tilde{x}|| = r$. Furthermore, the maximization problem $\max(u, G)$ is also well-posed and $\tilde{g}$ is the unique furthest point to u from G. We estimate

$$H(F, Y) = d(u, F) \leqslant ||u - \bar{x}|| \leqslant ||u - \tilde{x}|| + ||\tilde{x} - \bar{x}|| \leqslant 5r/4.$$

We next show that $Y \in \mathscr{M}_k$. Since

$$\begin{aligned} ||u - \tilde{g}|| &= r + ||\tilde{x} - \tilde{g}|| \\ &\geqslant r - ||\tilde{x} - \bar{x}|| + e(\bar{x}, G) \\ &> r - r/4 + \mu_F - r/4 \\ &= \mu_F + r/2, \end{aligned}$$

it follows that

$$\mu_Y \geqslant ||u - \tilde{g}|| > \mu_F + r/2.$$

Let $\{(y_n, g_n)\}$ with $y_n \in Y$ and $g_n \in G$ be a maximizing sequence. Then,

$$\liminf_n\ e(y_n, G) \geqslant \lim_n ||y_n - g_n|| = \mu_Y > \mu_F + r/2.$$

This implies that there exists some positive integer N_1 such that $y_n \notin F$ and so $y_n = u$ for all $n \geqslant N_1$. Hence,

$$\lim_n ||g_n - u|| = \lambda_Y \geqslant e(u, G)$$

and $\{g_n\}$ is a minimizing sequence for $\max(u, G)$. But the problem $\max(u, G)$ is well-posed, we conclude that (g_n) strongly converges to $\tilde{g}$. It is evident that $(u, \tilde{g})$ is the unique solution of the problem $\max(Y, G)$ and so $\max(Y, G)$ is well-posed. That is, $Y \in E_o^{\mathscr{A}}(G)$. The proof is complete. □

The following three corollaries are now direct consequences of Theorem 3.2.

Corollary 3.4. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively weakly compact, and bounded subset of X. Then $E_o(G)$ is a dense G_δ-subset of $\mathscr{B}_G(X)$.*

Corollary 3.5. *Suppose that X is a strictly convex and Kadec Banach space. Let G be a nonempty closed, relatively weakly compact, and bounded subset of X. Then $E_o(G) \cap \mathscr{K}(X)$ is a dense G_δ-subset of $\mathscr{K}(X)$.*

Corollary 3.6. *Suppose that X is a strongly convex Banach space and G is a nonempty closed bounded subset of X. Let $\mathscr{A}$ be an admissible family of $\mathscr{B}(X)$. Then $E_o^{\mathscr{A}}(G)$ is a dense G_δ-subset of $\mathscr{A}$.*

4. Porosity

The following definition is taken from De Blasi et al. [4].

Definition 4.1. A subset Y in a metric space (E, d) is said to be *porous* in E if there are $0 < t \leqslant 1$ and $r_0 > 0$ such that for every $x \in E$ and $r \in (0, r_0]$ there is a point $y \in E$ such that $B_d(y, tr) \subseteq B_d(x, r) \cap (E \backslash Y)$. A subset Y is said to be σ-porous in E if it is a countable union of sets which are porous in E.

Note that in this definition the statement "for every $x \in E$" can be replaced by "for every $x \in Y$". Clearly, a set which is σ-porous in E is also *meager* in E; the converse is, in general, false.

4.1. Minimization problems

For $F \in E^o(G)$, let (f_F, g_F) denote the unique solution to the problem $\min(F, G)$. Set

$$F_\alpha := F \cup \{(1-\alpha)f_F + \alpha g_F\}, \quad 0 \leqslant \alpha \leqslant 1.$$

Lemma 4.1. *Let $\mathbf{N} = \{1, 2, \ldots\}$ be the set of positive integers. Define a set $\tilde{\mathscr{B}}$ in $\mathscr{B}_G(X)$ by*

$$\tilde{\mathscr{B}} = \bigcap_{k \in \mathbf{N}} \bigcup_{F \in E^o(G)} \bigcup_{0 \leqslant \alpha \leqslant 1/2} B_{\mathscr{B}_G(X)}(F_\alpha, \rho_{F_\alpha}(1/k)),$$

where $\rho_{F_\alpha}(\varepsilon) = \min\{H(F, F_\alpha), 1\}\varepsilon$. If X is uniformly convex, then $\tilde{\mathscr{B}} \subseteq E^o(G)$.

Proof. It suffices to show that for every $F \in \tilde{\mathscr{B}}$,

$$\lim_{\delta \to 0+} \operatorname{diam} L_G(F, \delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0+} \operatorname{diam} L_F(G, \delta) = 0.$$

Indeed, as $F \in \tilde{\mathscr{B}}$, for each $k \in \mathbf{N}$, there exist $F^k \in E^o(G)$ and $0 \leqslant \alpha_k \leqslant 1/2$ such that

$$H(F, F^k_{\alpha_k}) \leqslant \rho_{F^k_{\alpha_k}}(1/k).$$

This implies that $H(F, F^k_{\alpha_k}) \to 0$ as $k \to +\infty$. It follows that

$$r_0 := \sup_{k \in N} \operatorname{diam} F^k_{\alpha_k} < +\infty.$$

We write for convenience,

$$\delta_k := \rho_{F^k_{\alpha_k}}(1/k), u_k := (1 - \alpha_k) f_{F^k} + \alpha_k g_{F^k},\ r_k := \lambda_{F^k}.$$

Then it is not hard to see that

$$\lambda_{F^k_{\alpha_k}} = (1 - \alpha_k) r_k,$$

$$d(u_k, F^k) = ||f_{F^k} - u_k|| = \alpha_k r_k,$$

$$\delta_k = \min\{H(F^k, F^k_{\alpha_k}), 1\}/k \leqslant \alpha_k r_k / k.$$

We may also assume, with no loss of generality, that $\alpha_k > 0$ for all k.

Claim I. $\operatorname{diam} L_F(G, \delta_k) \leqslant 4\delta_k$ *for all* $k > 4$.

To prove Claim I we first show

$$L_{F^k_{\alpha_k}}(G, 4\delta_k) = \{u_k\} \quad \forall k > 4.$$

To see this, we assume $f \in L_{F^k_{\alpha_k}}(G, 4\delta_k)$ and $k > 4$. Since

$$d(f, G) \leqslant (1 - \alpha_k) r_k + 4\delta_k \leqslant r_k - (1 - 4/k)\alpha_k r_k < \lambda_{F^k},$$

we obtain that $f \notin F^k$ and hence $f = u_k$ for $F^k_{\alpha_k} = F^k \cup \{u_k\}$.

Now for any $f \in L_F(G, \delta_k)$, since $H(F, F^k_{\alpha_k}) \leqslant \delta_k$, there exists $\bar{f} \in F^k_{\alpha_k}$ such that $||f - \bar{f}|| \leqslant 2\delta_k$. We have

$$d(\bar{f}, G) \leqslant ||f - \bar{f}|| + d(f, G) \leqslant \lambda_F + 3\delta_k \leqslant \lambda_{F^k_{\alpha_k}} + 4\delta_k,$$

using the fact that $|\lambda_F - \lambda_{F^k_{\alpha_k}}| \leqslant \delta_k$. It follows that $\bar{f} \in L_{F^k_{\alpha_k}}(G, 4\delta_k)$ and so $\bar{f} = u_k$ when $k > 4$. Thus, for any $f_1, f_2 \in L_F(G, \delta_k)$, we have that

$$||f_1 - f_2|| \leqslant ||f_1 - u_k|| + ||u_k - f_2|| \leqslant 4\delta_k \quad \forall k > 4,$$

This proves the claim.

Claim II. $L_G(F, \delta_k) \subseteq D(f_{F^k}, u_k, r_k, 4/k)$ *for all* $k > 4$.

To prove Claim II we first show

$$L_G(F^k_{\alpha_k}, 3\delta_k) \subseteq D(f_{F^k}, u_k, r_k, 4/k) \quad \forall k > 4.$$

In fact, for any $g \in L_G(F^k_{\alpha_k}, 3\delta_k)$, we have that

$$d(g, F^k_{\alpha_k}) \leqslant \lambda_{F^k_{\alpha_k}} + 3\delta_k = (1 - \alpha_k) r_k + 3\delta_k.$$

Take $f^k \in F^k_{\alpha_k}$ such that

$$||g - f^k|| \leqslant (1 - \alpha_k) r_k + 4\delta_k.$$

Hence for $k > 4$,

$$d(f^k, G) \leqslant ||g - f^k|| \leqslant (1 - \alpha_k) r_k + 4\delta_k \leqslant r_k - (1 - 4/k)\alpha_k r_k < \lambda_{F^k}.$$

This implies that $f^k = u_k$ and

$$\begin{aligned} ||g - u_k|| &\leqslant (1 - \alpha_k) r_k + 4\delta_k \\ &\leqslant (1 - \alpha_k) r_k + 4/k\alpha_k r_k \\ &= r_k - ||f_{F^k} - u_k||(1 - 4/k). \end{aligned}$$

It follows that

$$L_G(F^k_{\alpha_k}, 3\delta_k) \subseteq D(f_{F^k}, u_k, r_k, 4/k).$$

On the other hand, since

$$L_G(F, \delta_k) \subseteq L_G(F^k_{\alpha_k}, 3\delta_k),$$

we have

$$L_G(F, \delta_k) \subseteq D(f_{F^k}, u_k, r_k, 4/k) \quad \forall k > 4.$$

This ends the proof of Claim II.

Combining Claims I, II and Proposition 2.3, we have

$$\lim_{k \to +\infty} \operatorname{diam} L_G(F, \delta_k) = 0$$

and

$$\lim_{k \to +\infty} \operatorname{diam} L_F(G, \delta_k) = 0.$$

Hence by Proposition 2.2 $F \in E^o(G)$ and the proof is complete. $\square$

Theorem 4.1. *Let X be a uniformly convex Banach space and $\mathscr{A} \subseteq \mathscr{B}_G(X)$ be an admissible family of $\mathscr{B}(X)$. Then the set $\mathscr{A} \backslash E^o(G)$ is σ-porous in $\mathscr{A}$.*

Proof. Let

$$\mathscr{B}_k = \mathscr{A} \backslash \bigcup_{F \in E^o(G)} \bigcup_{0 \leqslant \alpha \leqslant 1/2} B_{\mathscr{A}}(F_\alpha, \rho_{F_\alpha}(1/k)),$$

$$\mathscr{B}_{kl} = \{F \in \mathscr{B}_k : 1/l < \lambda_F < l\}.$$

By Lemma 4.1, we have

$$\mathscr{A} \backslash E^o(G) \subseteq \mathscr{A} \backslash \tilde{\mathscr{B}} = \bigcup_{k \in \boldsymbol{N}} \bigcup_{l \in \boldsymbol{N}} \mathscr{B}_{kl}.$$

To complete the proof it suffices to show that the set $\mathscr{B}_{kl}$ is porous in $\mathscr{A}$ for every $k, l \in \boldsymbol{N}$.

Let $k, l \in \boldsymbol{N}$ be arbitrary. Define $r_0 = 1/(2l)$ and $\alpha = 1/(4k)$. By Theorem 3.1, for any $F \in \mathscr{B}_{kl}$ and $0 < r \leqslant r_0$, there exists $\bar{F} \in E^o_{\mathscr{A}}(G)$ such that

$$H(F, \bar{F}) < \frac{r}{4} \quad \text{and} \quad \frac{1}{l} < \lambda_{\bar{F}} < l.$$

Set $\bar{u}_{1/2} = (f_{\bar{F}} + g_{\bar{F}})/2$. Then

$$\begin{aligned} H(\bar{F}_{1/2}, F) &\geqslant H(\bar{F}_{1/2}, \bar{F}) - H(\bar{F}, F) \\ &\geqslant \sup_{f \in \bar{F}_{1/2}} d(f, \bar{F}) - r/4 \\ &\geqslant d(\bar{u}_{1/2}, \bar{F}) - r/4 \\ &= (1/2)\lambda_{\bar{F}} - r/4 \geqslant 3r/4. \end{aligned}$$

It follows that there exists $0 < t \leqslant 1/2$ such that $H(\bar{F}_t, F) = 3r/4$. Since for each $A \in B_{\mathscr{A}}(\bar{F}_t, \alpha r)$

$$H(A, F) \leqslant H(A, \bar{F}_t) + H(\bar{F}_t, F) \leqslant \alpha r + 3r/4 \leqslant r,$$

we have that

$$B_{\mathscr{A}}(\bar{F}_t, \alpha r) \subseteq B_{\mathscr{A}}(F, r).$$

In order to show that

$$B_{\mathscr{A}}(\bar{F}_t, \alpha r) \subseteq \mathscr{A} \backslash \mathscr{B}_{kl},$$

it suffices to show that

$$B_{\mathscr{A}}(\bar{F}_t, \alpha r) \subseteq B_{\mathscr{A}}(\bar{F}_t, \rho_{\bar{F}_t}(1/k)).$$

Indeed, from the definition of α, it follows that $\alpha r \leqslant 1/k$. Furthermore, since

$$H(\bar{F}_t, \bar{F}) \geqslant H(\bar{F}_t, F) - H(F, \bar{F}) \geqslant r/2,$$

we have

$$\alpha r \leqslant 2\alpha H(\bar{F}_t, \bar{F}) \leqslant H(\bar{F}_t, \bar{F})/k,$$

so that

$$\alpha r \leqslant \rho_{\bar{F}_t}(1/k).$$

This completes the proof. □

Corollary 4.1. *Let X be a uniformly convex Banach space. Then the set* $\mathscr{B}_G(X)\backslash E^o(G)$ *is* σ*-porous in* $\mathscr{B}_G(X)$.

Corollary 4.2. *Let X be a uniformly convex Banach space. Then the set* $\mathscr{K}_G(X)\backslash E^o(G)$ *is* σ*-porous in* $\mathscr{K}_G(X)$.

4.2. Maximization problems

Given $F \in E_o(G)$, let (f_F, g_F) be the unique solution to the problem $\max(F, G)$. Set

$$F_\alpha := F \cup \{(1+\alpha)f_F - \alpha g_F\}, \quad 0 \leqslant \alpha \leqslant 1.$$

We also set

$$\tilde{\mathscr{B}} := \bigcap_{k \in N} \bigcup_{F \in E_o(G)} \bigcup_{0 \leqslant \alpha \leqslant 1/2} B_{\mathscr{B}}(F_\alpha, \rho_\alpha(1/k)).$$

Lemma 4.2. *If X a is uniformly convex Banach space, then* $\tilde{\mathscr{B}} \subseteq E_o(G)$.

Proof. It suffices to show that, for every $F \in \tilde{\mathscr{B}}$,

$$\lim_{\delta \to 0+} \operatorname{diam} M_G(F, \delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0+} \operatorname{diam} M_F(G, \delta) = 0.$$

Let $F \in \tilde{\mathscr{B}}$ be arbitrary. Then for each $k \in N$ there exist $F^k \in \mathscr{B}_o$ and $0 \leqslant \alpha_k \leqslant \frac{1}{2}$ such that

$$H(F, F^k_{\alpha_k}) \leqslant \rho_{\alpha_k}(1/k).$$

So

$$r_0 := \sup_{k \in N} \operatorname{diam} F^k_{\alpha_k} < +\infty.$$

As in the proof of Lemma 4.1, we write

$$\delta_k := \rho_{\alpha_k}(1/k), \quad u_k := (1+\alpha_k)f_{F^k} - \alpha_k g_{F^k}, \quad r_k := \mu_{F^k}$$

and assume that $\alpha_k > 0$ for all k. Then we have

$$\delta_k \leqslant \alpha_k r_k / k, \quad \mu_{F^k_{\alpha_k}} = (1+\alpha_k) r_k$$

and

$$d(u_k, F^k) = ||f_{F^k} - u_k|| = \alpha_k r_k.$$

Let $\bar{r}_k := r_k + \alpha_k r_k (1 - 4/k)$. We will prove that

$$M_G(F, \delta_k) \subseteq M_G(F^k_{\alpha_k}, 3\delta_k) \subseteq D(u_k, f_{F^k}, \bar{r}_k, 4/k) \quad \forall k > 4 \tag{4.1}$$

and

$$M_{F^k_{\alpha_k}}(G, 4\delta_k) = \{u_k\} \quad \forall k > 4. \tag{4.2}$$

Firstly, for any $g \in M_G(F^k_{\alpha_k}, 3\delta_k)$, we have that

$$e(g, F^k_{\alpha_k}) \geqslant \mu_{F^k_{\alpha_k}} - 3\delta_k = (1+\alpha_k)r_k - 3\delta_k.$$

Take $f^k \in F^k_{\alpha_k}$ such that

$$\|g - f^k\| \geqslant (1+\alpha_k)r_k - 4\delta_k$$

so that for $k > 4$,

$$e(f^k, G) \geqslant \|g - f^k\| \geqslant (1+\alpha_k)r_k - 4\delta_k > r_k + (1-4/k)\alpha_k r_k > \mu_{F^k}.$$

This implies that $f^k = u_k$. Hence we have that

$$\begin{aligned} \|g - u_k\| &\geqslant (1+\alpha_k)r_k - 4\delta_k \\ &\geqslant (1+\alpha_k)r_k + 4\alpha_k r_k/r \\ &= \bar{r}_k. \end{aligned}$$

Clearly, $\|g - f_{F^k}\| \leqslant r_k = \bar{r}_k - \|u_k - f_{F^k}\|(1-4/k)$. This shows that

$$M_G(F^k_{\alpha_k}, 3\delta_k) \subseteq D(u_k, f_{F^k}, \bar{r}_k, 4/k)$$

since $g \in M_G(F^k_{\alpha_k}, 3\delta_k)$ is arbitrary. Noting that, for $H(F, F^k_{\alpha_k}) \leqslant \delta_k$,

$$M_G(F, \delta_k) \subseteq M_G(F^k_{\alpha_k}, 3\delta_k),$$

we have

$$M_G(F, \delta_k) \subseteq D(u_k, f_{F^k}, \bar{r}_k, 4/k);$$

hence (4.1) holds. Secondly, for any $f \in M_{F^k_{\alpha_k}}(G, 4\delta_k)$, we have

$$e(f, G) \geqslant \mu_{F^k_{\alpha_k}} - 4\delta_k = (1+\alpha_k)r_k - 4\delta_k \geqslant r_k + (1-4/k)\alpha_k r_k > r_k.$$

This implies that $f \notin F^k$ so that (4.2) holds. Next we will show that $F \in E_o(G)$. Indeed, for any $f \in M_F(G, \delta_k)$, since $H(F, F^k_{\alpha_k}) \leqslant \delta_k$, there exists $\bar{f} \in F^k_{\alpha_k}$ such that $\|f - \bar{f}\| \leqslant 2\delta_k$. Thus, we have that

$$e(\bar{f}, G) \geqslant e(f, G) - \|f - \bar{f}\| \geqslant \mu_F - 3\delta_k \geqslant \mu_{F^k_{\alpha_k}} - 4\delta_k,$$

which implies $\bar{f} \in M_{F^k_{\alpha_k}}(G, 4\delta_k)$, which, by (4.2), in turn implies that $\bar{f} = u_k$. Hence, for any two elements $f_1, f_2 \in M_F(G, \delta_k)$,

$$\|f_1 - f_2\| \leqslant \|f_1 - u_k\| + \|u_k - f_2\| \leqslant 4\delta_k.$$

Therefore,

$$\operatorname{diam} M_F(G, \delta_k) \leqslant 4\delta_k$$

and

$$\lim_{k\to+\infty} \operatorname{diam} M_F(G,\delta_k) = 0.$$

By Proposition 2.4 and (4.1), we conclude that

$$\lim_{k\to+\infty} \operatorname{diam} M_G(F,\delta_k) = 0.$$

This together with Proposition 2.2 indicates that $F \in E_o(G)$ and the proof is complete. □

Theorem 4.2. *Let X be a uniformly convex Banach space and $\mathscr{A}$ an admissible family of $\mathscr{B}(X)$. Then the set $\mathscr{A}\backslash E_o(G)$ is σ-porous in $\mathscr{A}$.*

Proof. Let

$$\mathscr{B}_k := \mathscr{A}\backslash \bigcup_{F\in E_o(G)} \bigcup_{0\leqslant\alpha\leqslant 1/2} B_{\mathscr{A}}(F_\alpha, \rho_{F_\alpha}(1/k)),$$

$$\mathscr{B}_{kl} := \{F\in\mathscr{B}_k :\ 1/l<\mu_F<l\}.$$

Then, by Lemma 4.1, we have

$$\mathscr{A}\backslash E_o(G)\subseteq\mathscr{A}\backslash\tilde{\mathscr{B}} = \bigcup_{k\in\boldsymbol{N}} \bigcup_{l\in\boldsymbol{N}} \mathscr{B}_{kl}.$$

By arguments similar to those used in the proof of Theorem 4.1, we can show that the set $\mathscr{B}_{kl}$ is porous in $\mathscr{A}$ for every $k,l\in\boldsymbol{N}$, and the proof is thus complete. □

Corollary 4.3. *Let X be a uniformly convex Banach space. Then the set $\mathscr{B}(X)\backslash E_o(G)$ is σ-porous in $\mathscr{B}(X)$.*

Corollary 4.4. *Let X be a uniformly convex Banach space. Then the set $\mathscr{K}(X)\backslash E_o(G)$ is σ-porous in $\mathscr{K}(X)$.*

5. Conclusions

We have established some results on generic property and porosity of well-posedness of mutually nearest and mutually furthest points for any admissible family of bounded subsets in Banach space. In particular, for a nonempty closed subset G of X, we obtained the following two results: one is this: if X is strongly convex, then $E^o(G)$ and $E^o(G)\cap\mathscr{K}(G)$ (resp. $E_o(G)$ and $E_o(G)\cap\mathscr{K}(G)$) are dense G_δ-sets in $\mathscr{B}_G(X)$ and $\mathscr{K}_G(X)$ (resp. $\mathscr{B}(X)$ and $\mathscr{K}(X)$), respectively; the other shows that $\mathscr{B}_G(X)\backslash E^o(G)$ and $\mathscr{K}_G(X)\backslash E^o(G)$ (resp. $\mathscr{B}(X)\backslash E_o(G)$ and $\mathscr{K}(X)\backslash E_o(G)$) are σ-porous in $\mathscr{B}_G(X)$ and $\mathscr{K}_G(X)$ (resp. $\mathscr{B}(X)$ and $\mathscr{K}(X)$), respectively, provided that X is uniformly convex. Recall that the first result was showed to be true for $\mathscr{K}^C_G(X)$ (resp. $\mathscr{K}^C(X)$) by Li [12] but for $\mathscr{B}^C_G(X)$ (resp. $\mathscr{B}^C(X)$) by De Blasi et al. [5] under the stronger assumption that X is uniformly convex. Here $\mathscr{B}^C(X)$ stands for the subset of $\mathscr{B}(X)$ consisting of all convex subsets of X and $\mathscr{B}_G^C(X)$, $\mathscr{K}^C(X)$, $\mathscr{K}_G^C(X)$ are defined similarly. However, the σ-porosity of the minimization problem $\min(A, G)$ (resp. maximization problem $\max(A, G)$) has not been explored for $A \in \mathscr{K}_G^C(X)$ (resp. $A \in \mathscr{K}^C(X)$) or $A \in \mathscr{B}_G^C(X)$ (resp. $A \in \mathscr{B}^C(X)$) before. Thus, we are motivated to consider the following two problems.

Problem 5.1. Is the set $E^o(G) \cap \mathscr{B}_G^C(X)$ (resp. $E_o(G) \cap \mathscr{B}^C(X)$) a dense G_δ-sets in $\mathscr{B}_G^C(X)$ (resp. $\mathscr{B}^C(X)$) if X is just a strongly convex Banach space?

Problem 5.2. Are the sets $\mathscr{B}_G^C(X) \backslash E^o(G)$ and $\mathscr{K}_G^C(X) \backslash E^o(G)$ (resp. $\mathscr{B}^C(X) \backslash E_o(G)$ and $\mathscr{K}^C(X) \backslash E_o(G)$) are σ-porous in $\mathscr{B}_G^C(X)$ and $\mathscr{K}_G^C(X)$ (resp. $\mathscr{B}^C(X)$ and $\mathscr{K}^C(X)$), respectively, if X is uniformly convex?

Surprisingly, the techniques developed in this paper or other papers such as [5,12] do not work for the above two problems and hence we leave them open.

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